On spectral span and eigenvalue-eigenvector assignment for a class of infinite dimensional control systems

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RESUME

On étudie le contrôle de systèmes distribués dont la dynamique génère un semi-groupe compact.

On montre d'abord que le spectre de tels systèmes est constitué d'une infinité de valeurs propres. De plus on montre que l'espace d'état est engendré par les vecteurs propres généralisés correspondants. Ensuite on étudie le problème du placement des valeurs propres et vecteurs propres.

ABSTRACT

This paper studies the infinite dimensional compact control systems which often happen in applications. First of all, it is shown that the spectrum of compact system consists of infinite set of eigenvalues. Furthermore, we prove that the state space is spanned by the generalized eigenvectors. Secondly, eigenvalue-eigenvector assignability is considered. The condition which realize the eigenvalue-eigenvector assignment is derived.

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1 - INTRODUCTION

This paper is concerned with the spectral span of the state space and eigenvalue-eigenvector assignment of a infinite dimensional control system.

If the state space is finite dimensional, then the spectrum of the state operator consists of a finite set of eigenvalues and the state space is spanned by its generalized eigenvectors. It is well knowm [11], [12] that every eigenvalue of the system can be shifted to any place in the complex plane by state feedback if and only if the system is controllable. Furthermore, we may assign generalized eigenvectors beyond eigenvalue selection [9], [10].

On the contrary, in the infinite dimensional case, the spectrum of a state operator is very chaotic even if it is a discrete operator [3]. FEINTUCH and ROSENFELT [6] considered the system whose state operator is discrete normal and proved that the controllable system can be reduced to a single input system while preserving controllability. Also they proved a pole assignability theorem. FEND, ZHU and HU [7] studied the infinite dimensional linear system with single input under the assumption that the state operator is scalar, spectral and discrete. They derived the formula of pole assignment which realize the infinite number of pole assignment and applied the result to the elastic vibration system for a space vehicle with slender body.

In this paper, we mainly consider the compact control system, that is, the system whose state operator generates a compact semigroup. In Section 3, we shall show that the state operator has infinite number of eigenvalues and the state space is spanned by its generalized eigenvectors. A condition which realize the eigenvalue-eigenvector assignment is dirived in Section 4. Some examples are given in Section 5.

2 - MATHEMATICAL PRELIMINARIES

Let $\mathscr{X}, \mathscr{U}$ be two separable complex Hilbert spaces. Consider the linear system

$$(2.1) \qquad \frac{d}{dt} u(t) = Au(t) + B\, f(t), \qquad t > 0$$

where $u \in \mathscr{X}$ (the state space), $F \in \mathscr{U}$ (the control space). The operator A is the infinitesimal generator of a C_0-semigroup $T(t)$, $t \geq 0$. Then A is closed with dense domain $\mathscr{D}(A)$ in $\mathscr{X}$. B is a bounded operator from $\mathscr{U}$ to $\mathscr{X}$ (for bervity we write $B \in \mathscr{L}(\mathscr{U};\mathscr{X})$). By the solution of (2.1) we mean a weak solution [1] of the form

$$(2.2) \qquad u(t) = T(t)\, u(0) + \int_0^t T(t-s)Bf(s)ds\ , \ t \geq 0$$

for $u(0) \in \mathscr{X}$, $f(.) \in L_2((0,T),\mathscr{U})$, T finite. The set of all states reachable in time $t > 0$ starting with zero initial state is denoted by $\mathscr{R}_t(A;B)$, that is,

$$\mathscr{R}_t(A;B) = \{u \in \mathscr{H} \mid u = \int_0^t T(t-s)Bf(s)ds,\ f \in L_2((0,t);\mathscr{U})\}.$$

Let

$$\mathscr{H}_c(A;,B) = \overline{\bigcup_{t\geq 0} \mathscr{R}_t(A;B)}$$

then $\mathscr{H}_c(A;B)$ is called a controllable subspace and the system is said to be controllable if $\mathscr{H}_c(A;B) = \mathscr{H}$. Let $f(t) = Fu(t)$ for $F \in \mathscr{L}(\mathscr{H};\mathscr{U})$, then the clased loop system

$$\frac{d}{dt}u(t) = (A + BF)\ u(t)$$

is called the state feedback system. It is well-known [4], [5] that

$$\mathscr{H}_c(A;B) = \overline{\bigcup_{t\geq 0} T(t) \text{ Range } B}$$

$$= \overline{\bigvee_{n\geq 0} R(\lambda;A)^n \text{ Range } B}$$

$$\text{for } \lambda > w_o = \lim_{t\to\infty} \log ||T(t)|| / t$$

where $R(\lambda; A) = [\lambda I - A]^{-1}$, i.e. the resolvent of A. As in the finite dimensional case,

$$\mathscr{H}_c(A,B) = \mathscr{H}_c(A+BF,B) \text{ for } \forall F \in \mathscr{L}(\mathscr{H};\mathscr{U})$$

that is the controllable subspace of $\{A;B\}$ coincide with the one of $\{A+BF,B\}$ [2].

DEFINITION 2.1.

An operator A is said to be discrete if its resolvent $R(\lambda;A) = [\lambda I - A]^{-1}$ is compact for all $\lambda \in \rho(A)$.

The following lemma is due to DUNFORD and SCHWARTZ [3].

LEMMA 2.1.

If A is discrete, then

a) A has pure point spectrum consisting at most of a countable sequence of points with no finite limit point.

b) every λ_o in $\sigma(A)$ is a pole of finite order $\nu(\lambda_o)$ of the resolvent.

c) for each positive integer n $\mathrm{Ker}([\lambda_o I-A]^n)$ is finite dimensional and

$\mathrm{Ker}([\lambda_o I-A]^m) = \mathrm{Ker}([\lambda_o I-A]^{m+1})$ for $m \geq \nu(\lambda_o)$, $\nu(\lambda_o)$ is the smallest positive integer with this property. ($\mathrm{Ker}([\lambda_o I-A]^{\nu(\lambda_o)})$ is called the space of generalized eigenvectors of A corresponding to λ_o.

d) if $P \in \mathcal{L}(\mathcal{H})$, then A+P is also discrete.

DEFINITION 2.2.

A C_o-semigroup T(t) is called compact if T(t) is compact for all $t > 0$. The system (2.1) is said to be compact if A is the infinitesimal generator of a compact semigroup T(t).

Many important properties of the compact semigroup have been studied in detail in [1]. We may state some of them in the form of a lemma.

LEMMA 2.2.

If A generates a compact semigroup T(t), then

a) A is discrete

b) $T(t)$ is uniformly continuous for $t \geq 0$

c) if P is a bounded operator, then $A+P$ generates a compact semigroup.

REMARK 2.1.

The fact that A is discrete doesn't imply in general that A generates a compact semigroup (see [1] for a counter example).

3 - SPECTRAL SPAN

This section studies the approximation by generalized eigenvectors. We begin with a basic definition and with introducing an equivalent class of vectors concerning with controllable subspaces.

DEFINITION 3.1.

Let A be a discrete operator in $\mathscr{H}$. Then the smallest closed subspace containing all the solution u of the equations $[\lambda I-A]^{\nu(\lambda)} u = 0$ with λ in $\sigma(A)$ is called the spectrol span of A and denoted by $\overline{sp}(A)$.

Let

$$[v] = \{u \in \mathscr{H} \mid \mathscr{H}_c(A;u) = \mathscr{H}_c(A;v)\}$$

for v in $\mathscr{H}$, then obviously $[v]$ is an equivalent class. The following lemmas will be used later.

LEMMA 3.1.

a) $[v] \subset \mathscr{H}_c(A;v)$

b) $R(\lambda;A)\,[v] \subset [v]$ for $\lambda \in \rho(A)$

c) there exists a $\delta > 0$ such that $S(v;\delta) \subset [v]$, if $v \neq 0$, where
$S(v;\delta) = \{u \in \mathscr{H}_c(A;v) \mid \; || u-v || \leq \delta\}$

d) furthermore, we can choose a $\delta' > 0$ such that if $u_n \in S(v;\delta')$ is a weakly convergent sequence and $u = \underset{n\to\infty}{\text{w-lim}}\, u_n$, then $u \in [v]$.

Proof

a) is obvious, since $u \in [v]$ implies $u \in \mathscr{H}_c(A;u) = \mathscr{H}_c(A;v)$.

For the proof of b) it is enough to show that $R(\lambda;A)v \in [v]$. Let $u = R(\lambda;A)v$, then $u \in \mathscr{D}(A)$ and $u \in \mathscr{H}_c(A;u) \subset \mathscr{H}_c(A;v)$. On the other hand $v = [\lambda I-A]u \in \mathscr{H}_c(A;u)$, since $A[\mathscr{H}_c(A;u) \cap \mathscr{D}(A)] \subset \mathscr{H}_c(A;u)$. This means $\mathscr{H}_c(A;v) \subset \mathscr{H}_c(A;u)$. Therefore $\mathscr{H}_c(A;u) = \mathscr{H}_c(A;v)$, i.e., $R(\lambda;A)v \in [v]$.

To prove c), first notice that $w \in \mathscr{H}_c(A;v)^{\perp}$ if and only if the inner product $(w,T(t)v)$ in $\mathscr{H}$ vanishes for all $t \geq 0$. Suppose for any $\delta > 0$, there exist vectors w_δ, $e_\delta \in \mathscr{H}_c(A;v)$ such that $||w_\delta|| = 1$, $0 < ||e_\delta|| < \delta$

$$(w_\delta,\ T(t)(v+e_\delta)) = 0 \text{ for all } t \geq 0.$$

Then

$$(3.1) \qquad |(w_\delta,\ T(t)v)| = |(w_\delta,\ T(t)e_\delta)| \leq ||T(t)||\delta$$

for all $t \geq 0$. This is the contradiction, since $\mathscr{H}_c(A;v) \cap \mathscr{H}_c(A;v)^{\perp} = \{0\}$ and (3.1) implies for any $\varepsilon > 0$, there exists an unit vector $w(\varepsilon)$ such that

$$(w(\varepsilon),u) < \varepsilon||u|| \quad \text{for all } u \text{ in } \mathscr{H}_c(A;v).$$

Suppose now for any $\delta' > 0$; there exists an unit vector $w \in \mathscr{H}_c(A;v)$, and a weakly convergent sequence $u_n \in S(v;\delta')$ such that

$$(w,T(t)u)_{\mathscr{H}} = 0 \text{ for all } t \geq 0, \text{ where } u = \underset{n\to\infty}{\text{w-lim}}\ u_n.$$

Then

$$\lim_{n\to\infty} |(w,\ T(t)u_n)| = 0 \text{ for any } t \geq 0.$$

Therefore, for any $t \geq 0$

$$|(w,T(t)v)| \leq |(w,T(t)u_n)| + |(w,T(t)(v-u_n))|$$

$$\leq |(w,T(t)u_n)| + ||(w,T(t)||\delta'$$

The first term tends to zero as $n \to \infty$ for any $t \geq 0$. This leads to the contradiction and proves d).

LEMMA 3.2.

If A generates a compact semigroup T(t), then the resolvent operator $R(\lambda;A)$ has at least one eigenvalue, that is, $R(\lambda;A)$ is never quasi-nilpotent.

Proof

Suppose $R(\lambda;A)$ has no eigenvalues. Let $\mathcal{M}_v$ be the controllable subspace of the pair $\{A,v\}$ for $v \neq 0$ in $\mathcal{H}$. Then $\mathcal{M}_v = \overline{\bigvee_{n\geq 0} R(\lambda;A)^n\{v\}}$ for $\lambda > w_o$ and is infinite dimensional, for otherwise $R(\lambda;A)|\mathcal{M}_v$ has an eigenvalue. By lemma 3.1 we may choose a $\delta > 0$ such that $S(v;\delta) = \{u \in \mu_v | \; ||u-v|| \leq \delta\} \subset [v]$ and any weakly limit in $S(v;\delta)$ belongs to $[v]$. Then from lemma 3.1 , b) , $\overline{R(\lambda;A)S(v;\delta)} \subset [v]$ and compact, since $R(\lambda;A)$ is compact.

Therefore, for each u in $\overline{R(\lambda;A)S(v;\delta)}$ there exist a continuous function h(t) and open neighborhood $\mathcal{N}_u$ of u in $\mathcal{M}_v$ such that

$$\left(\int_0^t h(s)\; T(t-s)ds\right)\mathcal{N}_u \subset S(v;\delta)$$

for some $t > 0$. The operator $\int_0^t h(s)\; T(t-s)ds$ is well-defined in the uniform operator topology and compact, since T(t) is compact.

Therefore, by compactness, there exist a finite set of continuous functions $h_1(t), h_2(t), \ldots, h_K(t)$ and times $t_1, t_2, \ldots, t_K$, such that for each u in

$\overline{R(\lambda;A)S(v;\delta)}$, $\int_0^{t_i} h_i(t)\, T(t_i-t)dt$ maps u into $S(v;\delta)$ for some $i \in \{1,2, \ldots, K\}$.

For brevity, we write

$$\int_0^{t_i} h_i(t)\, T(t_i-t)dt = T_i.$$

Then for each positive integer n we may find suitable indices $i(1), i(2), \ldots, i(n)$ such that

$$\Big(\prod_{m=1}^{n} T_{i(m)}\Big)R(\lambda;A)^n v$$

$$= T_{i(1)}R(\lambda;A)T_{i(2)}R(\lambda;A) \ \ldots \ T_{i(n)}R(\lambda;A)v \in S(v;\delta)$$

Now let $M = \max\{ \,||T_i|| \ | 1 \leq i \leq K\} < +\infty$, then

$$||\Big(\prod_{m=1}^{n} T_{i(m)}R(\lambda;A)^n v|| \leq M^n||R(\lambda;A)^n|| \ ||v||$$

Since $R(\lambda;A)$ is quasi-nilpotent, we obtain

$$M||R(\lambda;A)^n||^{1/n}||v||^{1/n} \to 0 \text{ as } n \to +\infty$$

Therefore

$$||\Big(\prod_{m=1}^{n} T_{i(m)}R(\lambda;A)^n v||^{1/n} \to 0 \text{ as } n \to +\infty. \tag{3.2}$$

On the other hand notice that $0 \notin \overline{R(\lambda;A)S(v;\delta)}$, $0 \notin S(v;\delta)$, since $0 \notin [v]$. Hence for some $\rho > 0$, we have

$$\rho < || \ \Big(\prod_{m=1}^{n} T_{i(m)}\Big)R(\lambda;A)^n v||$$

or

$$\rho^{1/n} < ||\Big(\prod_{m=1}^{n} T_{i(m)}\Big)R(\lambda;A)^n v||^{1/n} \tag{3.3}$$

Since $\lim_{n\to\infty} \rho^{1/n} = 1$, (3.2) and (3.3) give a contradiction. This completes the proof.

REMARK 3.1.

This lemma is an improved version of lemma 2.2 in a paper [8] given by the authors at a conference on "Analysis and Optimization of Systems", Versailles, FRANCE, 1982.

The next theorem is one of the main results of this section.

THEOREM 3.1.

If A generates a compact semigroup T(t) then $\sigma(R(\lambda;A))$ is an infinite set.

Proof

Suppose $\sigma(R(\lambda;A))$ is a finite set, say $\sigma(R(\lambda;A)) = \{0, \mu_1, \mu_2, \ldots, \mu_N\}$, $\mu_r \neq 0$, i = 1, 2, ..., N. Let $E(\mu_r;R(\lambda;A))$ be the spectral projection corresponding to μ_r of $R(\lambda;A)$, and let

$$E = \sum_{r=1}^{N} E(\mu_r;R(\lambda;A))$$

then E is compact since $E\mathcal{H}$ is finite dimensional and furthermore $(I-E)\mathcal{H}$ is closed. Therefore

$$\mathcal{H} = E\mathcal{H} \oplus (I-E)\mathcal{H} \tag{3.4}$$

and both $E\mathcal{H}$ and $(I-E)\mathcal{H}$ are $R(\lambda;A)$-invariant, for E commutes with $R(\lambda;A)$. The decomposition (3.4) means taht the restriction $R(\lambda;A) \mid (I-E)\mathcal{H}$ is quasi-nilpotent. This leads to the contradiction, since $R(\lambda;A)$ is never quasi-nilpotent by lemma 3.2.

COROLLARY 3.1.

If A generates a compact semigroup T(t), then $\sigma(A)$ is an infinite set.

Proof

This is evident, since there is one-to-one correspondence between $\sigma_p(A)$ and $\sigma_p(R(\lambda;A))$.

The following lemma is due to [3] and it is necessary for the proof of the spectral span theorem.

LEMMA 3.3.

Let A be an unbounded discrete operator in the complex Hilbert space $\mathcal{H}$ with spectrum $\{\lambda_i\}$, then

$$\overline{sp}\ \{A\} = \mathcal{S}_\infty(A^*)^\perp$$

where

$$\mathcal{S}_\infty\ (A^*) = \{u \in \mathcal{H} \mid E(\bar{\lambda}_i;A^*)u = 0,\ 1 \le i < +\infty\}$$

and $E(\bar{\lambda}_i;A^*)$ is the spectral projection corresponding to $\bar{\lambda}_i$ of A^*. Furthermore, the space $\mathcal{S}_\infty(A^*)$ either is infinite dimensional or consists only of zero.

The next spectral span theorem is the central result of this section.

THEOREM 3.2.

Suppose A generates a compact semigroup T(t) and $\sigma_p(A) = \{\lambda_i\}^\infty_{i=1}$, then

$$\overline{sp}(A) = \mathcal{H}.$$

Proof

By lemma 3.3., $\overline{sp}(A)^\perp = \mathcal{S}_\infty(A^*)$ and $\mathcal{S}_\infty(A^*)$ either is infinite dimensional or consists only of zero. Notice that $\mathcal{S}_\infty(A^*)$ is $R(\lambda;A)^*$-invariant and

$$\sigma_p(R(\lambda;A)^*) = \{\frac{1}{\bar{\lambda}-\bar{\lambda}_i}, i = 1, 2, \dots\}.$$

Suppose $\mathscr{S}_\infty(A^*)$ is infinite dimensional, then

$$\sigma(R(\lambda;A)^* \mid \mathscr{S}_\infty(A)^*)) = \{0\},$$

since $E(\frac{1}{\bar{\lambda}-\bar{\lambda}_i} ; R(\lambda;A)^*) = E(\bar{\lambda}_i;A^*)$.

However this is impossible by lemma 3.2.

4 - EIGENVALUE - EIGENVECTOR ASSIGNMENT

In this section we consider the eigenvalue-eigenvector assignment by state feedback. We have the following

THEOREM 4.1.

a) if $\mu \in \rho(A) \cap \sigma_p(A+BF)$, then $1 \in \sigma_p(FR(\mu;A)B)$

b) if $\mu \in \rho(A)$ and $1 \in \sigma_p(FR(\mu;A)B)$, then $\mu \in \sigma_p(A+BF)$

Proof

Suppose $\mu \in \rho(A) \cap \sigma_p(A+BF)$, then there exists a nonzero u in $\mathscr{H}$ such that $\mu u = (A+BF)u$. A simple calculation gives us

(4.1) $\quad u = R(\mu;A)BFu.$

By operating with F on (4.1), we have

$$Fu = FR(\mu;A)BFu$$

This implies $1 \in \sigma_p(FP(\mu;A)B)$ and Fu is its eigenvector in $\mathscr{U}$

On the other hand if $1 \in \sigma_p(FR(\mu;A)B)$ and let ξ be an corresponding eigenvector, then $\xi = FR(\mu;A)B\xi$. Let $u = R(\mu;A)B\xi$, then $u \in \mathscr{D}(A)$ and $Fu = \xi$. Therefore, $(\mu I-A) = B\xi = BFu$, that is, $(A+BF)u = \mu u$. This completes the proof.

THEOREM 4.2.

Let $\{\mu_i\}_{i=1}^{\infty}$ be a set of distinct complex numbers which has no finite limit point. Suppose $\{\mu_i\}_{i=1}^{\infty} \subset \rho(A)$ and

$$(4.2) \qquad \overline{\bigvee_{i=1}^{\infty} R(\mu_i;A)b_i} = \mathscr{X}$$

for $b_i \in$ Range B, $b_i \neq 0$. Then there exists an operator F from $\mathscr{X}$ into $\mathscr{U}$ such that

$$\sigma_p(A+BF) = \{\mu_i\}_{i=1}^{\infty}$$

and

$$\psi_i = R(\mu_i;A)b_i \,/\, ||R(\mu_i;A)b_i|| \qquad i = 1, 2, \ldots$$

are corresponding normal eigenvectors. (In general ψ_i, $i = 1, 2, \ldots$ are not orthogonal).

Furthermore, if $b_i = B\xi_i$, $\xi_i \in \mathscr{U}$ and

$$\sum_{i=1}^{\infty} \frac{||\xi_i||^2}{||R(\mu_i;A)b_i||^2} < +\infty ,$$

then F is bounded.

Proof

From the definition of ψ_i, we have

$$(\mu_i I-A)\psi_i = \frac{b_i}{||R(\mu_i;A)b_i||}$$

$$= \frac{B\xi_i}{||R(\mu_i;A)b_i||} \qquad \xi_i \in \mathscr{U}$$

Let us define F by

$$F\psi_i = \frac{\xi_i}{||R(\mu_i;A)b_i||} \qquad \text{for } i = 1, 2, \ldots$$

Then $[\mu_i I-(A+BF)]\ \psi_i = 0$

Suppose now

$$\left(\sum_{i=1}^{\infty} \frac{||\xi_i||^2}{||R(\mu_i;A)b_i||^2}\right) = M < +\infty$$

For each u in $\mathscr{H}$ we have an unique representation

$$u = \sum_{i=1}^{\infty} u_i\psi_i$$

and

$$M_1(\sum_i |u_i|^2)^{1/2} \le ||u|| \le M_2(\sum_i |u_i|^2)^{1/2}$$

for suitable chosen positive constants M_1 and M_2. Then

$$Fu = \sum_{i=1}^{\infty} u_i\ F\psi_i = \sum_{i=1}^{\infty} \frac{u_i\ \xi_i}{||R(\mu_i;A)b_i||}$$

Therefore

$$||Fu|| \le \left(\sum_{i=1}^{\infty} |u_i|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} \frac{||\xi_i||^2}{||R(\mu_i;A)b_i||^2}\right)^{1/2}$$

$$\le (M/M_1)\ ||u||.$$

This completes the proof.

REMAKR 4.1.

The condition (4.2) implies that the pair $\{A;B\}$ is controllable, since $R(\mu_i;A)b_i \in \mathscr{H}_c(A;B)$.

REMARK 4.2.

In theorem 4.2., there is a flexibility of the selection of b_i for the same $\{\mu_i\}_{i=1}^{\infty}$. Therefore, we may select eigenvectors beyond eigenvalue selection as in the finite dimensional case [9], [10].

COROLLARY 4.1.

Let $\{\mu_i\}_{i=1}^{\infty}$ be a set of distinct complex numbers which has no finite limit point.

Suppose $\{\mu_i\}_{i=1}^{\infty} \subset \rho(A)$, $\overline{\bigvee_{i=1}^{\infty} R(\mu_i;A)b_i} = \mathcal{H}$ for non zero $b_i \in$ Range B, and

$$\sum_{i=1}^{\infty} \frac{||\xi_i||^2}{||R(\mu_i;A)b_i||^2} < +\infty \quad \text{for } b_i = B\xi_i,\ \xi_i \in \mathcal{U}.$$

Define F as in the proof of theorem 4.2. Let $f_i = F^*\xi_i^*$ for $\xi_i^* \in \text{Ker}\,[I_{\mathcal{U}} - B^*R(\mu_i;A)^*F^*]$, $\xi_i^* \neq 0$, and let

$$\psi_i' = \frac{R(\mu_i;A)^* f_i}{||R(\mu_i;A) f_i||}$$

Then $\{\psi_i'\}_{i=1}^{\infty}$ is a set of normal eigenvectors of $A^* + F^*B^*$ and $\sigma_p(A^* + F^*B^*) = \{\bar{\mu}_i\}_{i=1}^{\infty}$. Moreover $\{\psi_i'\}_{i=1}^{\infty}$ is a basis of $\mathcal{H}$ and

(4.3) $\qquad (\psi_i, \psi_j') = 0$ if and only if $i \neq j$.

Proof

Notice that F is bounded and $\text{Ker}\,[I_{\mathcal{U}} - B^*R(\mu_i;A^*F^*]$ is not trivial since $\text{Ker}[I_{\mathcal{U}} - FR(\mu_i;A)B]$ $\quad \xi_i \neq 0$. From the definition of ψ_v

$$(\bar{\mu}_i - A^*)\psi_i' = \frac{f_i}{||R(\mu_i;A)^* f_i||}$$

On the other hand

$$
\begin{aligned}
F^{*}B^{*}\psi_i' &= F^{*}B^{*}R(\mu_i;A)^{*}f_i/||R(\mu_i;A)^{*}f_i|| \\
&= F^{*}B^{*}R(\mu_i;A)^{*}F^{*}\xi_i^{*}/||R(\mu_i;A)^{*}f_i|| \\
&= F^{*}\xi_i^{*}/||R(\mu_i;A)^{*}f_i|| \\
&= f_i/||R(\mu_i;A)^{*}f_i||
\end{aligned}
$$

Therefore

$$\bar{\mu}_i\psi_i' = (A^{*} + F^{*}B^{*})\psi_i'$$

For the proof of (4.3) it is enough to show that $(R(\mu_i;A)^{*}f_i,\ R(\mu_j;A)b_j) = 0$ for $i \neq j$.

$$
\begin{aligned}
(R(\mu_i;A)^{*}f_i,\ R(\mu_j;A)b_j) &= (f_i,\ R(\mu_i;A)R(\mu_j;A)b_j) \\
&= \frac{1}{\bar{\mu}_j-\bar{\mu}_i}\{(f_i,\ R(\mu_i;A)b_j) - f_i,\ R(\mu_j;A)b_j)\} \\
&= \frac{1}{\bar{\mu}_j-\bar{\mu}_i}\{(R(\mu_i;A)^{*}F^{*}\xi_i^{*},\ B\xi_j) - \\
&\qquad - (F^{*}\xi_i^{*},\ R(\mu_j;A)B\xi_j)\} \\
&= \frac{1}{\bar{\mu}_j-\bar{\mu}_i}\{(B^{*}R(\mu_i;A)^{*}F^{*}\xi_i^{*},\xi_j) - \\
&\qquad - (\xi_i^{*},\ FR(\mu_j;A)B\xi_j)\} \\
&= \frac{1}{\bar{\mu}_j-\bar{\mu}_i}\{(\xi_i^{*},\xi_j) - (\xi_i^{*},\xi_j)\} \\
&= 0 \text{ for } i \neq j.
\end{aligned}
$$

COROLLARY 4.2.

Let $\{\mu_i\}_{i=1}^{\infty}$ be a set of distinct complex numbers which has no finite limit point. Suppose

$\{\mu_i\}_{i=1}^{\infty} \subset \rho(A)$, $\overline{\bigvee_{i=1}^{\infty} R(\mu_i;A)b_i} = \mathscr{H}$ for non zero $b_i \in$ Range B. Let us **select** ϕ_n' **in** $\mathscr{H}_n^{\perp}$ such that $(\phi_n', R(\mu_n;A)b_n) = 1$, where

$$\mathscr{H}_n = \overline{\bigvee_{i \neq n} R(\mu_i;A)b_i}$$

Then

a) $\overline{\bigvee_{n=1}^{\infty} \phi_n'} = \mathscr{H}$

b) define F by $Fu = \sum_{i=1}^{\infty} (u, \phi_i')\xi_i$ for $b_i = B\xi_i$.

$\mathscr{D}(F) = \{u \in \mathscr{H} | \sum_{i=1}^{\infty} (u\ \phi_i')\xi_i$ is convergent$\}$.

Then $\sigma_p(A+BF) = \{\mu_i\}_{i=1}^{\infty}$ and $\{R(\mu_i;A)b_i\}_{i=1}^{\infty}$ are corresponding **eigenvectors**

c) furthermore, if $\sum_{i=1}^{\infty} \dfrac{||\xi_i||^2}{||R(\mu_i;A)b_i||^2} < +\infty$, then F is bounded and $\{\phi_n'\}_{i=1}$ **are** eigenvectors of $A^* + F^*B^*$ which correspond to $\{\bar{\mu}_i\}_{i=1}$.

<u>Proof</u>

Since the codimension of $\mathscr{H}_n$ is one, we can select ϕ_n uniquely in $\mathscr{H}_n^{\perp}$. Suppose $(u, \phi_n') = 0$ for $n = 1, 2, \ldots$, then for the representation $u = \sum_n u_n R(\mu_n;A)b_n$, we have $u_n = (u,\phi_n') = 0$. This proves a).

Note that $R(\mu_i;A)b_i \in \mathscr{D}(F)$ for $i = 1, 2, \ldots$ and $FR(\mu_i;A)b_i = \xi_i$. Therefore

$$(A+BF)R(\mu_i;A)b_i = \mu_i R(\mu_i;A)b_i - (\mu_i I-A)R(\mu_i;A)b_i + b_i$$

$$= \mu_i R(\mu_i;A)b_i$$

This proves b).

To prove c), first note that

$$\begin{aligned}\|\phi_n'\| &= \sup_{\|u\|=1} |(\phi_n', u)| \\ &= \sup_j |(\phi_n', R(\mu_j;A)b_j)|/\|R(\mu_j;A)b_j\| \\ &= |(\phi_n', R(\mu_n;A)b_n)|/\|R(\mu_n;A)b_n\| \\ &= 1/\|R(\mu_n;A)b_n\|\end{aligned}$$

Therefore, $\{\|R(\mu_n;A)\|\phi_n\}_{i=1}^{\infty}$ is a normal basis of $\mathscr{H}$, and there exists constants C_1, C_2 such that

$$C_1\|u\| \le (\sum_n |(u,\|R(\mu_n;A)\|\phi_n)|^2)^{1/2} \le C_2\|u\|$$

Suppose now

$$(\sum_{i=1}^{\infty} \frac{\|\xi_i\|^2}{\|R(\mu_i;A)b_i\|^2})^{1/2} = M < +\infty,$$

then

$$\begin{aligned}\|Fu\| &\le \sum_{i=1}^{\infty} |(u,\|R(\mu_i;A)\|\phi_i)| \frac{\|\xi_i\|}{\|R(\mu_i;A)b_i\|} \\ &\le (\sum_{i=1}^{\infty} |(u,\|R(\mu_i;A)\|\phi_i)|^2)^{1/2} (\sum_{i=1}^{\infty} \frac{\|\xi_i\|^2}{\|R(\mu_i;A)b_i\|^2} \\ &\le C_2 M\|u\|\end{aligned}$$

that is, F is a bounded operator.

Let us note that F is the same as in the proof of theorem 4.2., since $FR\mu_i b_i = \xi_i$. Let f_i be a vector in $\mathscr{H}$ as defined in Corollary 4.1., then

$R(\mu_n;A)^* f_n \in \mathscr{K}_n^\perp$. Therefore

$$\phi_n = \frac{R(\mu_n;A)^* f_n}{(R(\mu_n;A)^* f_i,\ R(\mu_i;A) b_n)}$$

since $\mathscr{K}_n$ has codimension one. This completes the proof.

THEOREM 4.3.

Suppose A+BF has eigenvalues $\{\mu_i\}_{i=1}^\infty$ of multiplicity one and $\{\mu_i\}_{i=1}^\infty \subset \rho(A)$. Let $\{\psi_i\}_{i=1}^\infty$ (or $\{\psi_i'\}_{i=1}^\infty$) is a set of corresponding normal eigenvectors of A+BF (or $A^* + F^* B^*$) which is a basis of $\mathscr{K}$. Then there exist vectors b_i in range B (or f_i in Range F^*) such that

$$\overline{\bigvee_{n\geq 1} R(\mu_i;A) b_i} = \mathscr{K} \ (\text{or } \overline{\bigvee_{n\geq 1} R(\mu_i;A)^* \bar{f}_i} = \mathscr{K})$$

$$\psi_i = R(\mu_i;A) b_i \quad (\text{or } \psi_i' = R(\mu_i;A)^* f_i).$$

Proof

By definition

$$\mu_i \psi_i = A\psi_i + BF\psi_i \quad (\text{or } \bar{\mu}_i\ \psi_i' = A^* \psi_i' + F^* B^* \psi_i').$$

Therefore

$$\psi_i = R(\mu_i;A)\ BF\psi_i \quad (\text{or } \psi_i' = R(\mu_i;A)^* F^* B^* \psi_i').$$

Let $b_i = BF\psi_i$ (or $f_i = F^* B^* \psi_i'$), then

$$\psi_i = R(\mu_i;A) b_i \quad (\text{or } \psi_i' = R(\mu_i;A)^* f_i)$$

and

$$\overline{\bigvee_{n\geq 1} R(\mu_i;A)b_i} = \mathscr{H} \ (\text{or } \overline{\bigvee_{n\geq 1} R(\mu_i;A)^* f_i} = \mathscr{H})$$

REMARK 4.3.

Let d_n be the distance from μ_n to $\sigma(A+BF) - \{\mu_n\}$. Suppose

$$\sum_{n=1}^{\infty} \frac{1}{d_n^2} < +\infty$$

then by theorem 2.7. and its corollary [3] all but a finite number of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of A have multiplicity one and furthermore

$$|\mu_n - \lambda_n| \leq K, \ n \geq K_1$$

for suitably chosen constants K_1 and L with $K_1 \geq 1$.

5 - EXAMPLES

In this section we consider the eigenvalue-eigenvector assignment problem in detail through some examples.

EXAMPLE 1

Let $\mathscr{H} = \ell_2$ (the space of square summable sequences). We use the notation $u = \{u_n\}$ for u in ℓ_2, i.e., $\sum_{n=1}^{\infty} |u_n|^2 < +\infty$. Let $\{\lambda_n\}$ be any sequence of complex numbers which has no finite limit point. Define

$$Au = \{\lambda_n u_n\}$$

$$\mathscr{D}(A) = \{u \in \ell_2 \mid \sum_{i=1}^{\infty} |\lambda_n u_n|^2 < +\infty\}$$

then $\sigma(A) = \{\lambda_n\}$ and the resolvent operator $R(\lambda;A)$ is

$$R(\lambda;A)u = \{\frac{u_n}{\lambda-\lambda_n}\} \quad \text{for} \quad \lambda \neq \lambda_n, \ n = 1, 2, \ldots$$

The operator A is discrete, since $\{\lambda_n\}$ has no finite limit point. If $-\infty < \mathrm{Re}.\lambda_n \le w < +\infty$, then A is an infinitesimal generator of a C_o-semigroup $T(t)$ such that

$$T(t)u = \{e^{\lambda_n t} u_n\}$$

$T(t)$ is compact if and only if $\limsup_n \mathrm{Re}\,\lambda_n = -\infty$. We consider eigenvalue-eigenvector assignment problem in two cases.

<u>Case 1</u> $\lambda_i \neq \lambda_j$ for $i \neq j$

Let $\mathscr{U} = \mathscr{C}$ (complex plane) and B be a bounded operator from $\mathscr{U}$ into $\mathscr{H}$ such that $B\xi = \xi b$ for $\xi \in \mathscr{C}$, $b = \{b_n\} \in \mathscr{H}(=\ell_2)$. Then $\{A,B\}$ is controllable if and only if $b_n \neq 0, n = 1, 2, \ldots$ Therefore, it is easy to show that

$$\overline{\bigvee_{i=1}^{\infty} R(\mu_i;A)b} = \mathscr{H}$$

for any set of distinct complex numbers $\{\mu_i\}_{i=1}^{\infty}$ with no finite limit point if $\{A, B\}$ is controllable. Then by theorem 4.2., there exists a operator F (which may not bounded) from $\mathscr{H}$ into $\mathscr{C}$ such that $\sigma_p(A+BF) = \{\mu_i\}$ and $R(\mu_i;A)b/||R(\mu_i;A)b||$ are corresponding eigenvectors. Moreover, if

$$\sum_i \frac{1}{||R(\mu_i;A)b||^2} = \sum_i \frac{1}{\sum_n \frac{|b_n|^2}{|\mu_i-\lambda_n|^2}} < +\infty ,$$

F is bounded, i.e., F is a linear funcitonal, then by Riesz representation theorem, there exists a vector f in $\mathscr{H}$ such that

$$Fu = (u,f),\ (R(\mu_i;A)b.f) = 1 \text{ for } i = 1, 2, \ldots$$

This case we have no flexibility about eigenvector assignment.

<u>Case 2</u> $\{\lambda_n\}_{n=1}^{\infty}$ have multiplicity 2 (that is, $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4 \neq \ \ldots$)

Let $\mathscr{U} = \mathscr{C}^2$ and B be a bounded operator from $\mathscr{U}$ into $\mathscr{H}$ such that

$$B\xi = \xi_1 b^1 + \xi_2 b^2 \text{ for } \xi = \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} \in \mathscr{U}(=\mathscr{C}^2),\ b^1, b^2 \in \mathscr{H}(=\ell_2).$$

Then {A,B} is controllable if and only if

$$\text{Rank of Mat.} \begin{pmatrix} b^1_{2n-1} & b^1_{2n} \\ b^2_{2n-1} & b^2_{2n} \end{pmatrix} = 2 \text{ for } b^2 = \{b^1_n\},\ b^2 = \{b^2_n\}$$

Therefore, if {A,B} is controllable

$$\overline{\bigvee_{i=1}^{\infty} R(\mu_i;A)\tilde{b}_i} = \mathscr{H}$$

for any set of distinct complex numbers $\{\mu_i\}_{i=1}^{\infty}$ with no finite limit point and

$$\tilde{b}_{2i-1} = \bar{b}^1$$

$$\tilde{b}_{2i} = \bar{b}^2$$

where $\bar{b}^1$ and $\bar{b}^2$ are any linear combination of b^1 and b^2 such that $\bar{b}^1$ and $\bar{b}^2$ are independent. This case we have the flexibility of the eigenvector assignment beyond eigenvalue selection. Furthermore if

$$\sum_i \frac{1}{||R(\mu_i;A)b_i||^2} < +\infty \quad ,$$

then there exists vectors f_1 and f_2 in $\mathscr{H}$ such that

$$Fu = \begin{pmatrix}(u,f_1) \\ (u,f_2)\end{pmatrix}$$

$$(R(\mu_{2i-1};A)\tilde{b}_{2i-1},f_1) = \xi_{11}, \quad (R(\mu_{2i-1};A)\tilde{b}_{2i-1},f_2) = \xi_{12}$$

$$(R(\mu_{2i};A)\tilde{b}_{2i},f_1) = \xi_{21} \quad , \quad (R(\mu_{2i};A)\tilde{b}_{2i},f_2) = \xi_{22}$$

for $\bar{b}^1 = \xi_{11}b_1 + \xi_{12}b_2$, $\bar{b}^2 = \xi_{21}b_1 + \xi_{22}b_2$

REMARK 5.1.

In both cases, if Range B has dimension more than the multiplicity **of eigenvalues,** then we have much more flexibility of eigenvector assignment.

REMARK 5.2.

In Case 1, f has the representation

$$f = \{f_n\}$$

$$f_n = \frac{\bar{\lambda}_n\bar{\sigma}_n}{\bar{b}_r} \Big/ \left(1 - \sum_{K=1}^{\infty} \bar{\sigma}_K\right)$$

where

$$\sigma_n = \left(1 - \frac{\lambda_n}{\mu_n}\right) \prod_{j\neq n} \frac{\left(1 - \frac{\lambda_n}{\lambda_j}\right)}{\left(1 - \frac{\lambda_n}{\lambda_j}\right)} \qquad n = 1, 2, \ldots$$

if the further conditions

$$\sum_K \frac{1}{|\mu_K|} < +\infty, \quad \sum_K \frac{|\lambda_K|^2|\sigma_K|^2}{|b_K|^2} < +\infty, \quad \sum_K \sigma_K < +\infty$$

$$\sum_K \sigma_k \neq 1, \quad \sum_K |\lambda_K - \mu_K| < +\infty$$

$$\sup_K \sum_{j\neq K} \frac{1}{|\mu_K - \lambda_j|^2} < +\infty$$

are satisfied. (see [7] for details).

REMARK 5.3.

If the semigroup T(t) is compact then we can stabilize the controllable system in any desirable order since $\limsup_n \operatorname{Re} \lambda_n = -\infty$ (see [8]).

Next we study the second order system, that is, the system whose state operator is second order (with a set of boundary conditions which may make it nonself adjoint).

EXAMPLE 2

Let $\mathcal{H} = L_2(0,1)$ and consider the unbounded operator $A = (\frac{d}{dx})^2$ with the boundary conditions

$$u(0) - k_0 u'(0) = 0,$$

$$u(1) - k_1 u'(1) = 0$$

where k_0 and k_1 are arbitrary complex numbers possibly infinite. ($k_0 = \infty$ and/or $k_1 = \infty$ imply the conditions $u'(0) = 0$ and $u'(1) = 0$, respectively).

Case 1 $\quad k_0 = k_1 = \infty$

In this case, A is densely defined selfadjoint operator and has a pure point spectrum with no finite limit point such that

$$\sigma(A) = \sigma_p(A) = \{\lambda \in \mathcal{C} \mid \lambda = \lambda_n = -(n-1)^2 \pi^2, \ n = 1, 2, \ldots\}$$

and

$$\Phi_n(x) = \sqrt{2} \cos (n-1) \pi x, \ n = 1, 2, \ldots$$

is a set of orthonormal eigenvectors.

Furthermore A generate a compact semigroup T(t) such that

$$T(t)u = \sum_{n=1}^{\infty} \exp[-(n-1)^2\pi^2 t]\,(u.\Phi_n)\Phi_n$$

where

$$(u,\Phi_n) = \int_0^1 u(\underset{\sim}{x})\,\Phi_n(x)\,dx$$

The resolvent operator $R(\lambda;A)$ is

$$R(\lambda;A)u = \sum_{n=1}^{\infty} \frac{1}{\lambda+(n-1)^2}(u,\Phi_n)\Phi_n, \text{ for } \lambda \in \rho(A)$$

Let $\mathscr{U}=\mathscr{C}$ and define $B \in \mathscr{L}(\mathscr{U};\mathscr{X})$ by

$$B\xi = \xi b \text{ for } \xi \in \mathscr{C}\ , b \in \mathscr{X}$$

Them the same argument as in Case 1 of the previous example shows that

$$\overline{\bigvee_{i=1}^{\infty} R(\mu_i;A)b} = \mathscr{X}$$

for any set of complex numbers $\{\mu_i\}_{i=1}^{\infty}$ with no finite limit point if $(b,\Phi_n) \neq 0$ for all n = 1, 2, ... Furthermore, if

$$\sum_i \left(1 / \sum_n \frac{|(b,\Phi_n)\Phi^2}{|\mu_i+(n-1)^2|^2}\right) < +\infty$$

then there exists a bounded operator F such that $\sigma_p(A+BF) = \{\mu_i\}_{i=1}^{\infty}$ and A+BF generates a compact semigroup.

<u>Case 2</u> $k_0k_1 \neq 0, k_0 \neq \infty, k_1 \neq \infty$

This case, $\lambda \in \sigma(A)$ if and only if $\lambda = -s^2$, where s is a root of the equation

$$\tan s = \frac{cs}{1+ds^2}, \quad c = k_1 - \qquad d = k_0k_1$$

The corresponding eigenvector is sin $s(t+\alpha)$, where α is an constant which satisfy $\tan s\alpha = k_0 s$. And we have an estimate for the distribution of eigen-vlaues such that

$$\lambda_n = -(n\pi)^2 - 2cd^{-1} + 0(n^{-1})$$

for sufficiently large n. Moreover only a finite set of λ_n can be multiple poles of $R(\lambda;A)$, and $\sum_i E(\lambda_i;A)u$ converges unconditionally in the topology of $L_2(J)$, where J is a compact sub-interval of (0,1), [3]. It is easy to show that A generates a compact semigroup T(t) since a finite set of λ_n can be multiple more than one and $\lambda_n \to -\infty$.

Let m be a maximal multiplicity of $\{\lambda_n\}_{n=1}^{\infty}$ and let $\mathcal{U} = \mathcal{C}^m$. Define $B \in \mathcal{L}(\mathcal{U};\mathcal{X})$ by

$$B\xi = \sum_{i=1}^{m} \xi_i b_i \quad \text{for } \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \in \mathcal{C}^m$$

Then {A,B} is controllable if and only if

$$\dim E(\lambda_i;A)\,\text{Range } B = \dim E(\lambda;A)\mathcal{X}$$

since dimension of $E(\lambda_i;A)\,\mathcal{X} \le m$ and only a finite set of λ_n can be multiple more than 1.

Therefore, if {A,B} is controllable, then

$$\overline{\bigvee_{K=1}^{\infty} \bigvee_{i=1}^{m_k} R(\mu_{ki};A)\tilde{b}_{ki}} = \mathcal{X}$$

for any set of distinct complex numbers $\{\mu_{ki}; i = 1, 2, \ldots m_k, k=1,2,\ldots\} \subset \rho(A)$ with no finilite limit point and

$$\tilde{b}_{ki} = \bar{b}_i \qquad i = 1, 2, \ldots, m_k, k = 1, 2, \ldots$$

where $\bar{b}_i$, $i = 1, 2, \ldots, m$ are any linear combination of b_i, $i = 1, 2, \ldots, m$ such that $\bar{b}_i$, $i = 1, 2, \ldots, m$ are independent each other. Thus we have the flexibility of the eigenvector assignment.

Furthermore if $\sum_{k=1}^{\infty} \sum_{i=1}^{m_k} \frac{1}{||R(\mu_{ki};A)\tilde{b}_{ki}||^2} < +\infty$, there exists vectors f_{ki}, $i = 1, 2, \ldots, m_k$, $k = 1, 2, \ldots$, such that

$$(R(\mu_{ki};A)\tilde{b}_{ki}, f_{kj}) = \xi_{ij} \qquad \begin{array}{l} i = 1, 2, \ldots, m_k, \ j = 1, 2, \ldots, m_k \\ K = 1, 2, \ldots \end{array}$$

for

$$\bar{b}_i = \sum_{j=1}^{m} \xi_{ij} b_j$$

6 - CONCLUSION

This paper has considered the problem of spectral span and of eigenvalue-eigenvector assignment for a class of infinite dimensional control systems. We showed that the spectrum of compact system consists of infinite set of eigenvalues and the state space is spanned by its generalized eigenvectors. This fact makes us possible to classify the infinite dimensional discrete systems. Although the controllability of the system doesn't always imply the possibility of infinite number of pole assignment, we also derived the condition which realize the eigenvalue-eigenvector assignment and examined it through some examples.

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